Bull. Kagoshima Pref. Jr. College, **41**, 1 − 4 (1990)

Pushout Squares in The Category of Topological Spaces

By Kentaro Ozeki

(Received September 17, 1990)

We denote by Top the category of topological spaces and continuous maps.

A following diagram 1 in Top satisfies the conditions :

(1) $g' \circ f = f' \circ g$

(2) For any space T and any map $h : Y \longrightarrow T$, $k : Z \longrightarrow T$ with $h \circ f = k \circ g$ there exists a unique map $p : W \longrightarrow T$ such that $h = p \circ g'$ and $k = p \circ f'$.

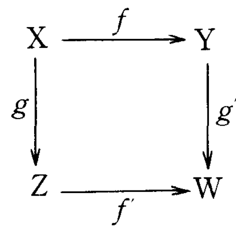

diagram 1

Then we say that diagram 1 is a pushout square. Note that space W is defined as a quotient space $(Y \amalg Z) / R$ where $Y \amalg Z$ is coproduct (disjoint union) Y and Z, R the equivalence relation generated by a relation ~ by setting $f(x) \sim g(x)$ on each $x \in X$. Also note that f, g are composites $Y \xrightarrow{i_Y} Y \amalg Z \xrightarrow{p} W$, $Z \xrightarrow{i_Z} Y \amalg Z \xrightarrow{p} W$ respectively where i_Y, i_Z are natural injections and p is natural projection. Let $Y \cup_f X$ be the adjunction space by a map $f : A \longrightarrow Y$ where A is a subset of X.

Then diagram 2 is a pushout square where $i : A \longrightarrow X$ is inclusion map.

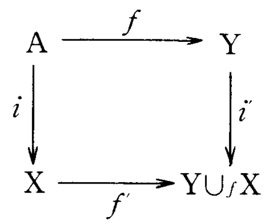

diagram 2

We are concerned with pushout square in Top.

1. We shall constantly use following proposition which is well known hence the proof is omitted.

Proposition 1.1 *In the commutative diagram 3 in Top, let the left hand square is a pushout. Then the right hand square is a pushout if and only if the exterior rectangle is a pushout.*

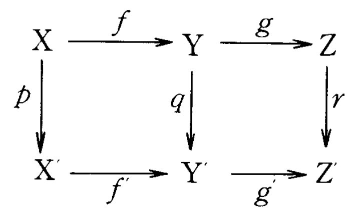

diagram 3

Proposition 1.2 *In the following pushout square in Top,*

1) *if f is a injection then so is f.*

2) *if f is a surjection then so is f.*

3) *if f is an identification then so is f.*

4) *if f is a closed map then so is f.*

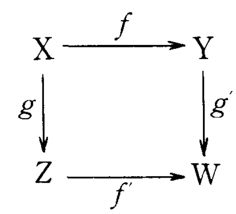

diagram 4

Proof. 1) Let z, z′ be elements of Z and suppose $f'(z) = f'(z')$. We shall show that $z = z'$.

Recall that f' is composites $Z \xrightarrow{i_z} Y \amalg Z \xrightarrow{p} W = (Y \amalg Z)/R$ where i_z is the natural injection and p is a natural projection, and R is equivalence relation generated by relation $\sim$ setting by $f(x) \sim g(x)$ for each $x \in X$. Then we have $p \circ i_z(z) = p \circ i_z(z')$ and $i_z(z) = i_z(z')$. Hence we have $z = z'$

2) Let w be element of W. Then there exists a $t \in Y \amalg Z$ such that $p(t) = w$. If $t \in Y$ then there exists a $x \in X$ such that $f(x) = t$ because f is a surjection. We have $f(x) = t \sim g(x)$. Then we have $f'(g(x)) = p \circ i_z(g(x)) = p(t) = w$.

3) Let T is a space in Top and $h: W \longrightarrow T$ be a map. Let $h \circ f'$ be continuous. Then we have $h \circ f' \circ g = h \circ g' \circ f$ and $h \circ g'$ is continuous map because f is a identification. h is continuous map, therefore W has final topology with respect to g'.

4) Let C is a closed subset of Z. We shall show that $f'(C)$ is a closed subse of W. To see this we need only show that $f'^{-1} \circ f'(C)$ is a closed subset of Z and $g'^{-1} \circ f'(C)$ is a closed subset of Y.

Now we have $f'^{-1} \circ f'(C) = C$ and $g'^{-1} \circ f'(C) = f \circ g^{-1}(C)$. Since the map f is closed, $f \circ g^{-1}(C)$ is closed subset of Y. This completes the proof.

Proposition 1.3 *Let X, Y, Z be spaces and $f: A \longrightarrow Y$, $g: Y \longrightarrow Z$ in Top where A is a subspace of X.*

Then we have a homeomorphism

$\varphi: Z \cup_g (Y \cup_f X) \cong Z \cup_{g \circ f} X$

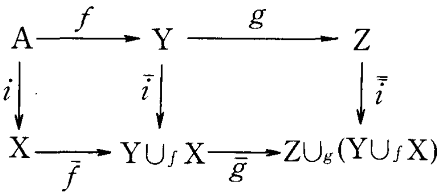

diagram5

Proof. In the diagram 5 left hand square and right hand one is pushout. Then exterior rectangle is pushout by proposition 1.1. It follows the universality of pushout.

Similarly we have the following.

Proposition 1.4 *Let X, Y be spaces and A, B subspaces of X with $A \subset B$ and map $f: A \longrightarrow Y$ in Top.*

Then we have homeomrphism $\varphi: (Y \cup_f B) \cup_{\bar{f}} X \cong Y \cup_f X$.

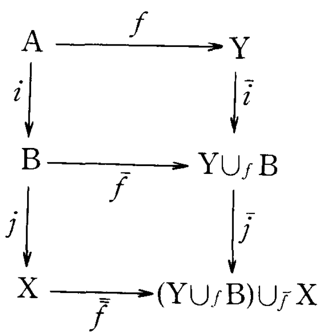

diagram 6

Let X, Y, Z be spaces and $f: X \longrightarrow Y$, $g: Y \longrightarrow Z$ be maps in Top.

Then we have diagram 7 where CX is cone over X and I is interval [0, 1] of real line R.

By the proposition 1.3 and 1.4 we have $Z \cup (Y \cup_f CX) \cong Z \cup_{g \circ f} CX$ and $(Y \cup_f CX) \cup_{\bar{f}} (X \times I) \cong Y \cup_f (X \times I)$.

We also have $(Z\cup_g (Y\cup_f CX))\cup_{\bar g}((Y\cup_f CX)\cup_{\bar f}(X\times I))\cong Z\cup_g((Y\cup_f CX)\cup_{\bar f}(X\times I))\cong Z\cup_g(Y\cup_f(X\times I))\cong Z\cup_{g\circ f}(X\times I)$.

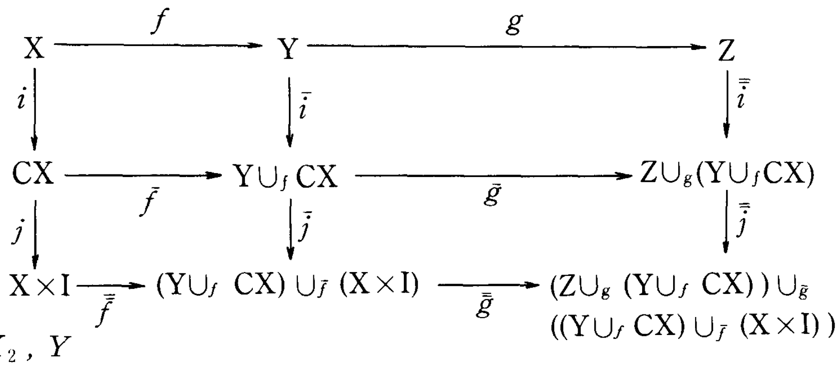

diagram 7

Proposition 1.5 *Let* X_1 , X_2 , Y *be spaces and* $A_1\subset X_1$, $A_2\subset X_2$ *be subspaces. Then there is a homeomorphism* $\varphi : X_2\cup_{i_2}(Y\cup_{f_1}X)\cong Y\cup_{f_1\amalg f_2}(X_1\amalg X_2)$.

Proof. By the conditions we have three pushout squares in diagram 8 where $i_1\amalg i_2$, $f_1\amalg f_2$ are induced by inclusions $i_1 : A_1\longrightarrow X_1$, $i_2 : A_2\longrightarrow X_2$ and $f_1 : A_1\longrightarrow Y$, $f_2 : A_2\longrightarrow Y$ respectively.

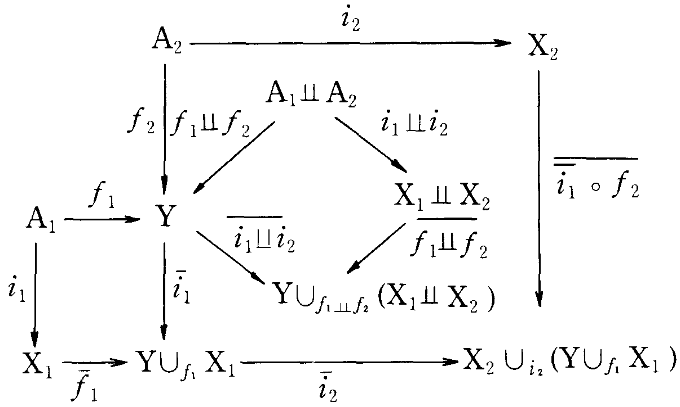

diagram 8

We have $\overline{i_1\amalg i_2}\circ f=\overline{f_1\amalg f_2}\circ k_2\circ \bar i_2$.

Then there exists a unique map $g : Y\cup_{f_1}X_1\longrightarrow Y\cup_{f_1\amalg f_2}(X_1\amalg X_2)$ such that $i_1\amalg i_2=g\circ i_1$ and $\overline{f_1\amalg f_2}\circ k_1=g\circ\bar f_1$.

We also have $\overline{f_1\amalg f_2}\circ k_2\circ i_2=g\circ\bar{\bar i}_1\circ f_2$.

Then there exists a unique map $\varphi : X_2\cup_{i_2}(Y\cup_{f_1}X)\longrightarrow Y\cup_{f_1\amalg f_2}(X_1\amalg X_2)$ such that $\overline{f_1\amalg f_2}\circ k=\varphi\circ\bar{\bar i}_1\circ f_2$ and $g=\varphi\circ\bar i_2$.

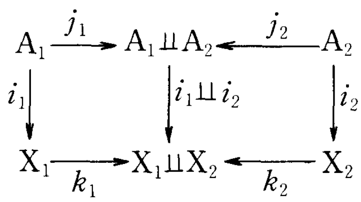

diagram 9

Let $h : X_1\amalg X_2\longrightarrow X\cup_{i_2}(Y\cup_{f_1}X_1)$ be a map induced by the maps $\bar i_2\circ f_1 : X_1\longrightarrow X_2\cup_{i_2}(Y\cup_{f_1}X_1)$ and $\bar{\bar i}_1\circ f_2 : X_2\longrightarrow X_2\cup_{f_2}(Y\cup_{f_1}X)$ Then we have $h\circ(i_1\amalg i_2)=\bar i_2\circ\bar i_1\circ(f_1\amalg f_2)$.

Hence there exists a unique map $\psi : Y\cup_{f_1\amalg f_2}(X_1\amalg X_2)\longrightarrow X_2\cup_{i_2}(Y\cup_{f_1}X_1)$ with $h=\varphi\circ\overline{f_1\amalg f_2}$ and $\bar i_2\circ\bar i_1=\varphi\circ\overline{i_1\amalg i_2}$. By the universality of pushout square $\psi\circ\varphi=$ identity on the space $X_2\cup_{i_2}(Y\cup_{f_1}X_1)$ and $\varphi\circ\psi=$ identity on the space $Y\cup_{f_1\amalg f_2}(X_1\amalg X_2)$.

This completes the proof.

2. Next we shall show another application of pushout squares. We begin by proving the following proposition.

Proposition 2.1 *Let X and Y be spaces in Top and let $f: X \longrightarrow Y$ be a map (may be not coninuous). Let A and B be subsets of X such that $A \cup B = X$, $A - B \subset Int\ A$* and *$B - A \subset Int\ B$. If $f \mid A$, $f \mid B$ are continuous, so is f where $f \mid A$, $f \mid B$ are restriction of f on A, B respectively.*

Proof. Let $x \in X$ and U be a neighborhood of f (x) in Y. If $x \in A \cap B$ then there exist neighborhoods M, N of x in X such that $(f \mid A)^{-1}(U) = M \cap A$ and $(f \mid B)^{-1}(U) = M \cap B$. We have $M \cap N \subset (M \cap A) \cup (N \cap B) = f^{-1}(U)$.

If $x \in A - B$, then we have $A - B \subset \mathrm{Int}\ A$ by hypothesis. Hence subset A is a neighborhoos of x in X. Since $f \mid A$ is continuous, there exists a neighborhood M of x in X such that $(f \mid A)^{-1}(U) = M \cap A$.

We have $M \cap A = f^{-1}(U) \cap A \subset f^{-1}(U)$.

If $x \in B - A$, then we may show that $f^{-1}(U)$ is a neighborhood of x in X.

Proposition 2.2 *Let X be a space in Top and let A, B be subsets of X with $B \subset A$. If B is closed in X and $B \subset Int\ A$ then the following diagram* 10 *is a pushout square.*

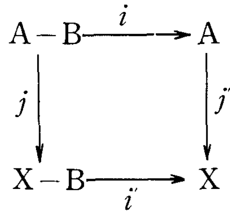

diagram 10

Proof. Let Y be space in Top and let maps $f: A \longrightarrow Y$, $g: X - B \longrightarrow Y$ be in Top with $f \circ i = g \circ j$.

We define a map $p: X \longrightarrow Y$ by $p(x) = f(x)$, $x \in A$ and $p(x) = g(x)$, $x \in X - A$. It is that $p \mid A$, $p \mid X - B$ are continuous.

We have $(X - B) - A \subset X - B$ and $A - (X - B) \subset \mathrm{Int}\ A$. By proposition 2.1 p is continuous and uniqueness of p is easily checked. This completes the proof.

Proposition 2.3 *With the notations and hypotheses of proposition* 2.2, *there exists a homeomorphism $h: (X - B) / (A - B) \cong X / A$.*

Proof. In the diagram 11, the left hand square is pushout by proposition 2.2 and also is right hand one.

By proposition 1.1 the exterior rectangle is a pushout. The assertion follows at once from the universality of pushout square. (We denots by X/A the quotient space of X with A identified to a point and ∗ is the one point space in Top.)

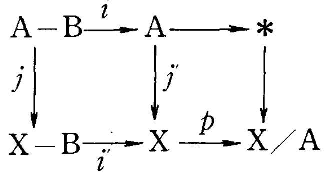

diagram 11

References

1) *R. Brown, Elements of Modern Topology, McGraw-Hill* (1968)

2) J. Dugundji, Topology, Ally and Bacon (1975)

3) S. MacLane, Categories for the Working Mathematician, Springer (1971)