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Pushout Squares in Top

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We denote by Top the category of topological spaces and continuous maps.

A following diagram 1 in Top satisfies the conditions:

(1) $g' \circ f = f' \circ g$

(2) For any space T and any maps $h : Y \longrightarrow T$, $k : Z \longrightarrow T$ with $h \circ f = k \circ g$ there exists a unique map $p : W \longrightarrow T$ such that $h = p \circ g'$ and $k = p \circ f'$.

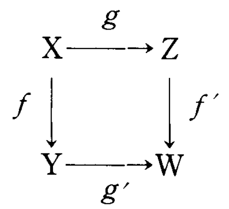

diagram 1

Then we say that diagram 1 is a pushout square. Note that space W is defined as a quotient space $(Y \amalg Z)/R$ where $Y \amalg Z$ is coproduct (disjoint union) Y and Z, R the equivalence relation generated by a relation $\sim$ by setting $f(x) \sim g(x)$ for each $x \in X$. Also note that g', f' are composites $Y \xrightarrow{i_Y} Y \amalg Z \xrightarrow{p} W$, $Z \xrightarrow{i_Z} Y \amalg Z \xrightarrow{p} W$ respectively where i_Y, i_Z are natural injections and p is the natural projection.

For example, let X be in Top and A be a subset of X.

We denote by X/A the quotient space of X with A identified to a point. Then the diagram 2 is a pushout square where $*$ is the one point space. Further let $Y \cup_f X$ be the adjunction space by a map $f : A \longrightarrow Y$ where A is a subset of X. Then the diagram 3 is a pushout square where $i : A \subset X$.

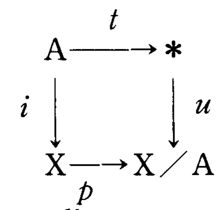

diagram 2

We are concerned with pushout squares in Top.

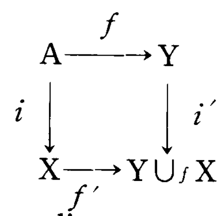

diagram 3

1. We shall constantly use the following proposition which at least is well known hence the proof is omitted.

Proposition 1.1 *In the commutative diagram 4 in Top, let the left square be a push out. Then the right square is a pushout if and only if the exterior rectangle is a pushout.*

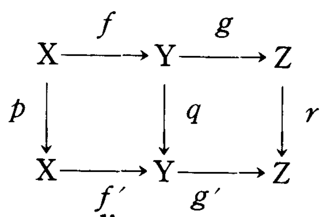

diagram 4

Proposition 1.2 *In the following pushout square in Top,*

1) *if f is a injection then so is f'.*

2) *if f is a closed map then so is f'.*

Proof. 1) Let z, z' be elements of Z and suppose $f'(z) = f'(z')$. We shall show that $z = z'$.

$$\begin{array}{ccc} X & \xrightarrow{g} & Z \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Y & \xrightarrow[g']{} & W \end{array}$$

diagram 5

Recall that f is composite $Z \xrightarrow{i_z} Y \amalg Z \xrightarrow{p} W = (Y \amalg Z)/R$ where i_Z is the natural injection, p is the natural projection and R is equivalence relation generated by relation $\sim$ setting $f(x) \sim g(x)$ for each $x \in X$.

Then we have $p \circ i_z(z) = p \circ i_z(z')$ and $i_z(z) = i_z(z')$. Hence we have $z = z'$.

3) Let C be a closed subset of Z. We shall show that $f'(C)$ is a closed subset of W.

To see this we need only show that $f'^{-1}_{\circ} f'(C)$ is a closed subset of Z and $g'^{-1}_{\circ} f'(C)$ is a closed subset of Y. Now we have $f'^{-1}_{\circ} f'(C) = C$ and $g'^{-1}_{\circ} f'(C) = f \circ g^{-1}(C)$. Since the map f is closed, $f \circ g^{-1}(C)$ the closed subset of Y. This completes the proof.

Proposition 1.3 *Let X and Y be in Top and let $f : X \longrightarrow Y$ be a set map (may be not continuous). Let A and B be subsets of X such that $A \cup B = X$, $A - B \subset \mathrm{Int}\, A$ and $B - A \subset \mathrm{Int}\, B$.*

If $f \mid A$ and $f \mid B$ are continuous, so is f, where $f \mid A$, $f \mid B$, are restriction of f on A, B respectively.

Proof. Let $x \in X$ and U be a neighborhood of $f(x)$ in Y. If $x \in A \cap B$ then there exist neighborhoods M, N of x in X such that $(f \mid A)^{-1}(U) = M \cap A$ and $(f \mid B)^{-1}(U) = N \cap B$.

We have $M \cap N \subset (M \cap A) \cup (N \cap B) = f^{-1}(U)$.

If $x \in A - B$, then we have $A - B \subset \mathrm{Int}\, A$ by hypothesis. Hence subset A is a neighborhood of x. Since $f \mid A$ is continuous, there exists a neighbourhood M of x in X such that

$(f \mid A)^{-1}(U) = M \cap A$. We have $M \cap A = f^{-1}(U) \cap A \subset f^{-1}(U)$.

If $x \in B - A$, then we can show that $f^{-1}(U)$ is a neighborhood of x in X, similarly.

2. Next we prove what is generally known as the theorems of adjunction space. We begin with a special case.

Proposition 2.1 *Let X be a space in Top and A, B is subsets of X with $A \subset B$. If subset B is closed in X there exists an embedding $j' : B/A \longrightarrow X/A$.*

Proof. In the diagram 6, the exterior rectangle is a pushout square where * is the one point space and those maps in the diagram 6 are inclusions and natural projections. By the proposition 1.1 the lower square is a pushout. Since j is injective and closed j' is an embedding by proposition 1.2.

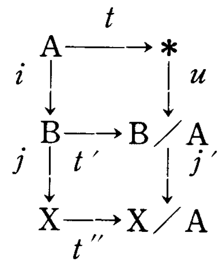

diagram 6

Lemma 2.2 *Let X be in Top and let A, B be subsets of X with $B \subset A$. If B is closed in X and $B \subset Int\ A$ then the following diagram 7 is a pushout square.*

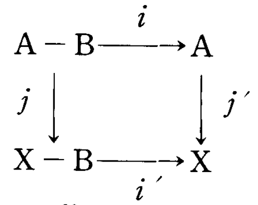

diagram 7

Proof. Let Y be in Top and let maps $f : A \longrightarrow Y$, $g : X - B \longrightarrow Y$ be in Top with $f \circ i = g \circ j$

We define the map $p : X \longrightarrow Y$ by $p(x) = f(x)$, $x \in A$ and $p(x) = g(x)$, $x \in X - A$.

It is obvious that $p \mid A$, $p \mid X - B$ are continuous.

We have $(X - B) - A \subset X - B$ and $A - (X - B) \subset \mathrm{Int} A$.

Hence p is continuous by proposition 1.3 and the uniqueness of p is easily checked.

This completes the proof.

Proposition 2.3 *With the notation and hypotheses of lemma 2.2, there exists a homeomorphism* $h : (X - B)/(A - B) \cong X/A$.

Proof. In the diagram 8, the left square is pushout by lemma 2.2 and also is the right square.

By proposition 1.1 the exterior rectangle is a pushout. The assertion follows at once from the definition of a pushout square.

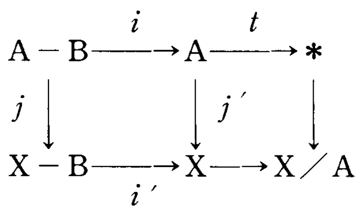

diagram 8

Proposition 2.4 *Let X be in Top and A be a subset of X and C A be the cone over A.*

Then there exists a homeomorphism $h : X \cup_i CA / CA \cong X/A$, *where those maps in the diagram 9 are inclusions and natural projection.*

Proof. In the diagram 9, the upper square is a pushout and is also lower one. Therefore the exterior rectangle is a pushout. The assertion follows immediately.

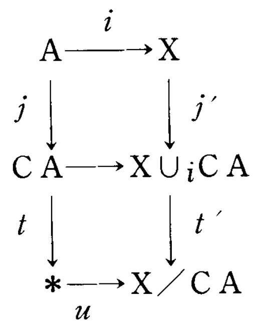

diagram 9

Let X, Y be spaces and let $f : X \longrightarrow Y$ be a map in Top. Then we obtain the following commutative diagram 10 where those maps are inclusions and natural projections.

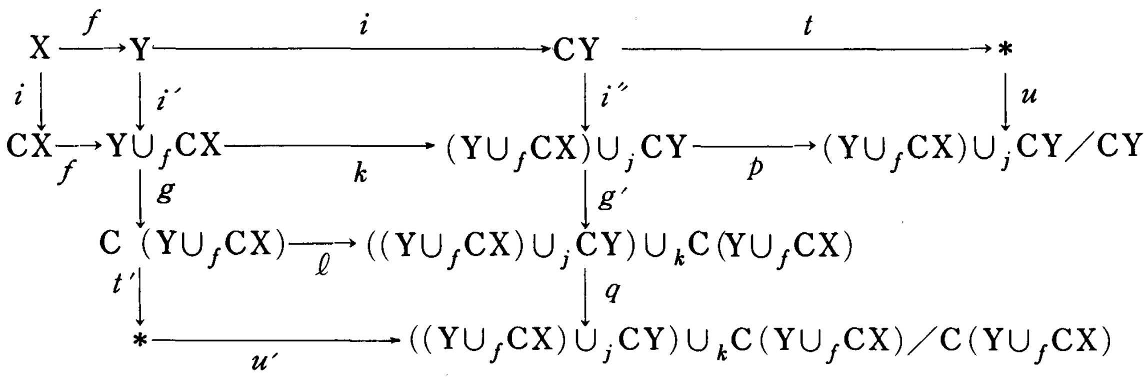

diagram 10

Since each square is a pushout, by proposition 1.1 and proposition 2.3 we have

$$
\begin{aligned}
(Y\cup_f CX)\cup_j CY/CY &\cong Y\cup_f CX/Y\\
&\cong CX/X\\
&\cong SX
\end{aligned}
$$

and

$$
\begin{aligned}
((CY\cup_f CX)\cup_j CY)\cup_k C(Y\cup_f CX)/C(Y\cup_f CX) &\cong (Y\cup_f CX)\cup_j CY/Y\cup_f CX\\
&\cong CY/Y\\
&\cong SY
\end{aligned}
$$

where SX is the suspension of X.

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