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Pushouts in The Category of Topological Spaces And Continuous Maps

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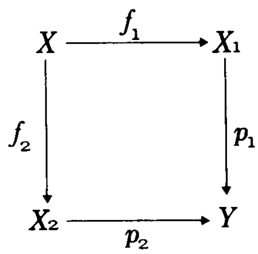
We denote by *Top* the category of topological spaces and continuous maps.

A following diagram in *Top* satisfies the conditions ;

(P1) $p_1 \circ f_1 = p_2 \circ f_2$

(P2) for any T in *Top* and any maps $f : X_1 \longrightarrow T$, $g : X_2 \longrightarrow T$ in *Top* with $f \circ f_1 = g \circ f_2$, there exists a unique map $h : Y \longrightarrow T$ such that $f = h \circ p_1$ and $g = h \circ p_2$.



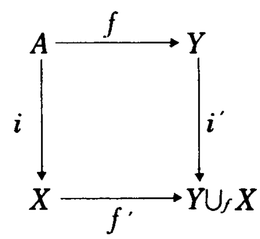
Then we say that (p_1, p_2) is the pushout of (f_1, f_2).

We construct the pushout as follows :

Let $f_1 : X \longrightarrow X_1$, $f_2 : X \longrightarrow X_2$ be in *Top*, and $X_1 \amalg X_2$ the disjoint union (coproduct) of X_1 and X_2 . We define a relation $\sim$ on this space by setting $f_1(x) \sim f_2(x)$ for each $x \in X$. We denote by R the equivalence relation generated by the relation $\sim$.

Let Y denote the space of equivalence classes $(X_1 \amalg X_2)/R$ and $p : X_1 \amalg X_2 \longrightarrow Y$ the natural projection.

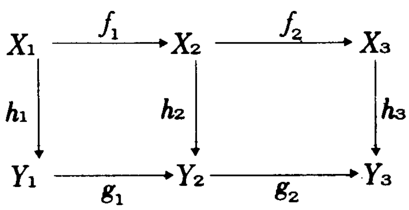
Let p_1, p_2 be the composites $X_1 \xrightarrow{i_1} X_1 \amalg X_2 \xrightarrow{p} Y$, $X_1 \xrightarrow{i_2} X_1 \amalg X_2 \xrightarrow{p} Y$ respectively where $i_1 : X_1 \longrightarrow X_1 \amalg X_2$, $i_2 : X_2 \longrightarrow X_1 \amalg X_2$ are natural injections. It is easy to see that (p_1, p_2) is the pushout of (f_1, f_2). In paticular, given spaces X, Y, a subspace A of X, inclusion map $i : A \longrightarrow X$ and a map $f : A \longrightarrow Y$, we denote by (i', f') the pushout of (f, i). This is well known as the adjunction space $Y \cup_f X$ by the map f.



We wish to deal with the some properties of pushouts in *Top*.

1. We shall begin by proving the following proposition which at least is well known. The proof is easy although it is tedious.

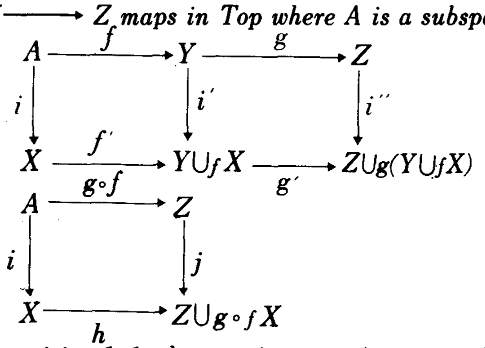
Proposition 1.1 *In the following commutative diagram in Top if both squares are pushouts then outside rectangle is a pushout.*



Proof. Let $f : X_3 \longrightarrow Z$, $g : Y_1 \longrightarrow Z$ be in *Top* with $f\circ f_2\circ f_1 = g\circ h_1$ We shall show that there exists a unique map $q : Y_3 \longrightarrow Z$ such that $f = q\circ h_3$ and $g = q\circ g_2\circ g_1$. Since the left square is the pushout, there exists a unique map $p : Y_2 \longrightarrow Z$ such that $f\circ f_2 = p\circ h_2$ and $g = p\circ g_1$. We also have $f = q\circ h_3$ and $p = q\circ g_2$ for a unique map $q : Y_3 \longrightarrow Z$, because the right square is the pushout. Then we have $f = q\circ h_3$ and $g = q\circ g_2\circ g_1$. The uniqueness of q is easily checked. This completes the proof of the proposition.

We shall now give some applications of this proposition.

Proposition 1.2 *Let X, Y, Z be spaces and $f : A \longrightarrow Y$, $g : Y \longrightarrow Z$ maps in Top where A is a subspace of X.*



Let (i', f') be the pushout of (f, i) and (i'', g') the pushout of (g, i').

Let (j, h) denote the pushout of $(g\circ f, i)$.

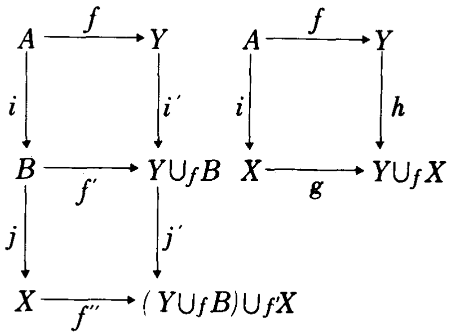
Then we have homeomorphism $\varphi : Z\cup_g(Y\cup_f X) \cong Z\cup_{g\circ f}X$.

Proof. Since $(i'', g'\circ f')$ is the pushout of $(g\circ f, i)$ by proposition 1.1, there exists a unique map $\varphi : Z\cup_g(Y\cup_f X) \longrightarrow Z\cup_{g\circ f} X$ such that $h = \varphi\circ g'\circ f'$ and $j = \varphi\circ i''$.

Similarly there exists a unique map $\psi : Z\cup_{g\circ f}X \longrightarrow Z\cup_g(Y\cup_f X)$ with $i'' = \psi\circ j$ and $g'\circ f' = \psi\circ h$.

By the condition (P2) we have $\psi\circ\varphi$ = identity map on space $Z\cup_g(Y\cup_f X)$ and $\varphi\circ\psi$ = identity map on space $Z\cup_{g\circ f}X$.

In the same way, we have the following.



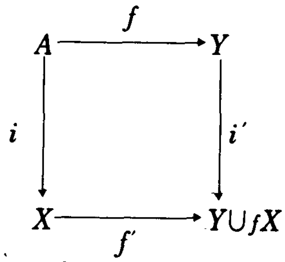
Proposition 1.3 *For given spaces X, Y and A, B subspaces of X with $A\subset B$ and map $f : A \longrightarrow Y$ in Top, (i', f') denotes the pushout of (f, i) and (j', f'') the pushout of (f', j) where i, j are inclusion maps $i : A \longrightarrow B$, $j : B \longrightarrow X$.*

Let (h, g) be the pushout of (f, i).

Then we have the homeomorphism $\varphi : (Y\cup_f B)\cup_{f'} X \cong Y\cup_f X$.

2. By using the pushout square we discuss the normality for adjucntion space.

Proposition 2.1 *For given spaces X, Y, a subset A is closed in X, and $f : A \longrightarrow Y$ is a map in Top. If the spaces X and Y are normal, then the adjunction space $Y\cup_f X$ by the map f is normal.*



Proof. Let (i', f') denote the pushout of (f, i) where i is the inclusion map $i : A \longrightarrow X$. Suppose that C is a closed set in $Y\cup_f X$ and a continuous map $h : C \longrightarrow I$ is given, where I is closed unit interval $[0, 1]$.

We shall show that there exists a map $h' : Y\cup_f X \longrightarrow I$ such that $h' \mid C = h$.

Let $i'^{-1}(C) = B$ and g be the composite $B \xrightarrow{i'} C \xrightarrow{h} I$. By the Tietzes extension theorem, we can extend g to a map $g' : Y \longrightarrow I$.

Let k be the composite $A \xrightarrow{f} Y \xrightarrow{g'} I$. We also have a map $k' : X \longrightarrow I$ such that $k' \mid A = k$. Since $k' \circ i = g' \circ f$, there exists a unique map $h' : Y \cup_f X \longrightarrow I$ such that $k' = h' \circ f$ and $g' = h \circ i'$.

It is easily checked that $h' \mid C = h$.

This completes the proof.

References

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