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A NOTE OF THE INITIAL OBJECT IN FUNCTOR CATEGORY

K. OZEKI

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1. Introduction

Let B, C be category, $F:B \longrightarrow C$ be functor and c be an object of C.

Then we define a category $(c \downarrow F)$ with objects (b,f) with $b \in B$ and $f:c \longrightarrow Fc$ in C.

A morphism in $(c \downarrow F)$ is a $h:(b,f) \longrightarrow (b',f')$ with $h:b \longrightarrow b'$ and $F(h)\ f=f'$. For given categories B and C, we may construct a functor category C^B with objects the functors $F:B \longrightarrow C$ and morphisms the natural transformations between to such functors.

We define a functor $\triangle :C \longrightarrow C^B$ by $\triangle(c)(b)=c$, $\triangle(c)(f)=1c$, and $\triangle(g)(b)=g$ for all $c \in C$, $b \in B$, $f \in B$ and $g \in C$, and we call $\triangle$ the diagonal functor.

Let A,B and C be categories. Let C^A be the functor category and $\triangle :C \longrightarrow C^A$ be the diagonal functor.

We define a functor

$$\bar{\triangle}: C^B \longrightarrow (C^A)^B$$

by $\bar{\triangle}(F) = \triangle \circ F$ for all $F \in C^B$.

We want to study the existence of the initial object in category $(T \downarrow \triangle)$ where $T \in (C^A)^B$.

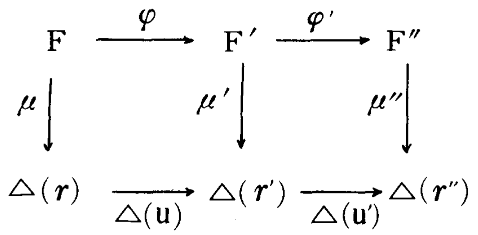
2. Preliminary remarks

Let A,B be categories, $F,F',F'':A \longrightarrow B$ be functors, $\varphi :F \longrightarrow F'$, $\varphi':F' \longrightarrow F''$ be natural transformations, and $\triangle:B \longrightarrow B^A$ be the diagonal functor.

Let (r, μ), (r', μ') and (r'', μ'') be initial objects in categories $(F\downarrow\triangle)$, $(F'\downarrow\triangle)$, $(F''\downarrow\triangle)$, respectively.

With these notations we obtain the following proposition.

Proposition 1 Natural transformations φ and φ' determine uniquely morphisms $u:r \longrightarrow r'$ and $u':r: \longrightarrow r''$ respectively such that following diagram is commutative;



Proof. The assertion is direct consequence of definition of the initial object.

If there exists the initial object in category $(F\downarrow\triangle)$ to each $F\in B^A$, then we may define a functor from B^A to category B by proposition 1.

Proposition 2 Let A,B be categories, $F:A\longrightarrow B$ be functor, and objects r,b be given. If there exists a bijection

$$\tau : A(r,a)\cong B(b, Fa)$$

such that τ is natural in a, then category $(b\downarrow F)$ has the initial object.

Proof is ommited (See S. Mac Lane 1)

3. Main therem

Let A,B and C be categories, and $\triangle:C\longrightarrow C^A$ be the diagonal functor.

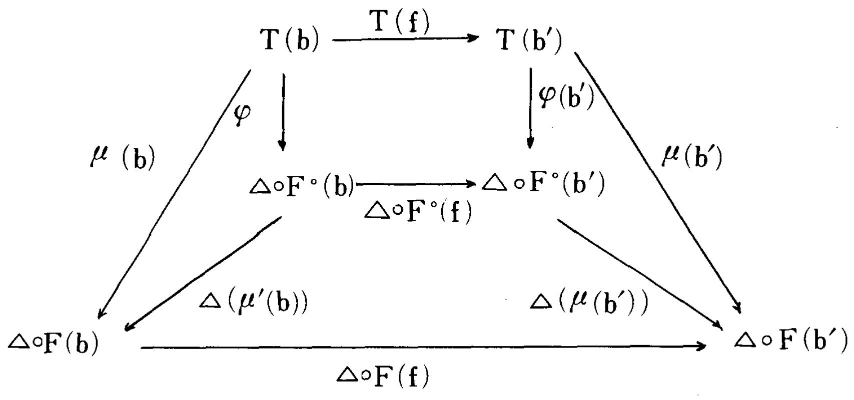
We define a functor

$$\bar{\triangle}:C^B\longrightarrow (C^A)^B$$

by $\bar{\triangle}(F)=\triangle\circ F$ for all $F\in C^B$.

For given $T\in(C^A)^B$ we assume that category $(T(b)\downarrow\triangle)$ has the initial object to each $b\in B$.

Let $F\in C^B$ and $\mu:T\longrightarrow\bar{\triangle}(F)$ be natural transformation. Since there exists the initial object $(F^{\circ}(b),\varphi(b))$ in category $(T(b)\downarrow\triangle)$ to each $b\in B$, we have the following diagram;



where $f: b \longrightarrow b'$ in B.

Since μ is a natural transformation and because of the property of the initial object, the diagram is commutative.

Then we have

$$\begin{aligned}\triangle \circ F(f) \circ \triangle(\mu'(b))\varphi(b) &= \triangle \circ F(f) \circ \mu(b)\\ &= F(b')T(f)\\ &= \triangle(\mu'(b'))\,\varphi(b')T(f)\\ &= \triangle(\mu'(b'))\triangle \circ F^{\circ}(f)\varphi(b).\end{aligned}$$

Hence $\triangle(F(f)\,\mu'(b))\varphi(b) = \triangle(\mu'(b')F^{\circ}(f))\,\varphi(b)$.

From the uniqueness of the initial object we have

$$F(f)\mu'(b) = \mu'(b')\,F^{\circ}(f)$$

It shows that $\mu' : F^{\circ} \longrightarrow F$ is a natural transformation.

We define a map

$$\tau : C^B(F^{\circ}, F) \longrightarrow (C^A)^B(T, \bar{\triangle}(F))$$

by $\tau(\psi) = (\bar{\triangle}\psi)\circ\varphi$ to each $\psi \in C^B(F^{\circ}, F)$

It is trivial to verify that τ is bijective and natural in F.

Then we have the following;

Theorem Let A,B and C be categories, $\triangle : C \longrightarrow C^A$ be the diagonal functor and $\triangle : C^B \longrightarrow (C^A)^B$ be the functor defined by $\bar{\triangle}(F) = \triangle \circ F$ for all $F \in C^B$.

Let $T \in (C^A)^B$.

If the category $(T(b)\downarrow\triangle)$ has the initial object to each $b \in B$, then there exists a bijection

$$\tau : C^B(F^{\circ}, F) \cong (C^A)^B(T, \bar{\triangle}(F))$$

where $F^{\circ} \in C^B$, that is, category $(T \downarrow \bar{\triangle})$ has the initial object.

Reference

1. S. Mac. Lane Categories for the working mathematician. Graduate texts in mathematics 5(1971)